

Base-40 arithmetic for music apps

Music 253/CS 275A

Stanford University

Where did Base-40 come from?

- Conceived by Walter Hewlett (1986); first pub 1992
- *Goals:* enharmonic spelling preservation, correct analysis, correct transposition
- Reproduced at <http://www.ccarh.org/publications/reprints/>
- Further elaborated in U.S. Patent 5,675,100 (7 October 1997)
<http://www.google.com/patents/US5675100>

Common bases in musical arithmetic

Subdivisions of the octave

- Base 7 (diatonic)
- Base 12 (semi-chromatic; MIDI)—favors eq-temp **sound**
- Base 21 (fully chromatic through 1 $\#/b$)—favors simple **notation**
- ?????? (19, 35.....)
- Base 40 (fully chromatic through 2 $\#/b$; supports invertible intervals for **analysis**)

Why Base-40?

- Musical literacy
- Tonal legibility (common-practice era)
- Musical computation in **integer arithmetic**
- Intervallic **complementarity**

Base-10 **complementarity**:

If interval = 3, complement = 7

If interval = 6, complement = 4

Review: Interval sizes and qualities

Prime 2nd 3rd 4th 5th 6th 7th 8th 9th 10th 11th 12th 13th 14th 15th

A musical staff in treble clef showing intervals from Prime to 15th. The notes are: Prime (C), 2nd (D), 3rd (E), 4th (F), 5th (G), 6th (A), 7th (B), 8th (C), 9th (D), 10th (E), 11th (F), 12th (G), 13th (A), 14th (B), and 15th (C).

M2 M3 M6 M7 aug 2nd aug 6th

A musical staff in treble clef showing major intervals: M2 (D), M3 (E), M6 (A), M7 (B), aug 2nd (D#), and aug 6th (A#).

Mi2 Mi3 Mi6 Mi7 dim 3rd dim 7th

A musical staff in treble clef showing minor intervals: Mi2 (Db), Mi3 (Eb), Mi6 (Ab), Mi7 (Bb), dim 3rd (Eb), and dim 7th (Bb).

P1 P4 P5 aug 1 aug 4th aug 5th dim 5th dim 5th

A musical staff in treble clef showing perfect and augmented/diminished intervals: P1 (C), P4 (F), P5 (G), aug 1 (C#), aug 4th (F#), aug 5th (G#), dim 5th (Fb), and dim 5th (Gb).

Review: Intervallic complementarity

M2 + m7 = 8ve M3 + m6 = 8ve P4 + P5 = 8ve P5 + P4 = 8ve M6 + m3 = 8ve M7 + m2 = 8ve

m2 + M7 = 8ve m3 + M6 = 8ve dim4 + aug5 = 8ve dim5 + aug4 = 8ve m6 + M3 = 8ve m7 + M2 = 8ve

aug2 + dim7 aug3 + dim6 aug4 + dim5 aug5 + dim4 aug6 + dim3

Review: Intervallic complementarity in chords

Major Minor Augmented Diminished

The diagram shows four chord types on a treble clef staff. Each chord is represented by a vertical stack of notes with arrows indicating the intervals between them. Major: m3, M3, P5. Minor: M3, m3, P5. Augmented: M3, #M3, aug5. Diminished: m3, m3, b9, dim5.

Root position First inversion Second inversion

Root → 5
Figured bass → 6 6 4

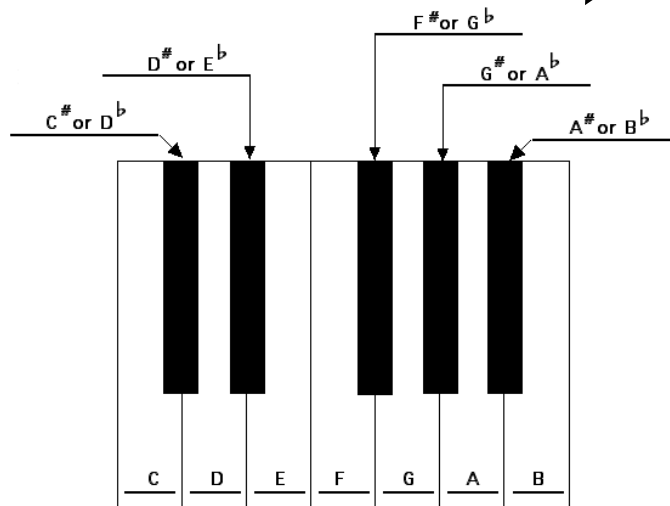
The diagram shows three chord positions on a treble clef staff. Root position: Root (C), 5 (G). First inversion: 6 (C), 6 (E). Second inversion: 4 (C), 6 (E), 4 (G). Arrows indicate the mapping from the root position to the inversions.

Integer arithmetic in digital analysis

- *Binomial solutions*: Brinkman, Böker-Heil
 - Required 3 params (pitch name, octave number, inflection)
- *Arbitrary mappings*: C=10, D=20, E=30....
 - Same-sized intervals do not always produced same numbers (depends on endpoints: F-E = 10, Eb-D = 9)
- Base-40 is **interval-invariant**:
 - it produces consistent arithmetical results
 - irrespective of endpoints and without binomials

Enharmonic-notation tiers

□ Physical instrument



©Enc

□ Cultural apparatus

- *Letter names*
 - Base-7 (0 #s/bs)
- *Octave numbers*
 - Base-12 (1#/b)
- *Inflection names*
 - Base-21 (1#/b)
- *Inflection names*
 - Base-40 (2#/b)

Wider system: Enharmonic-notation tiers

□ Third tier

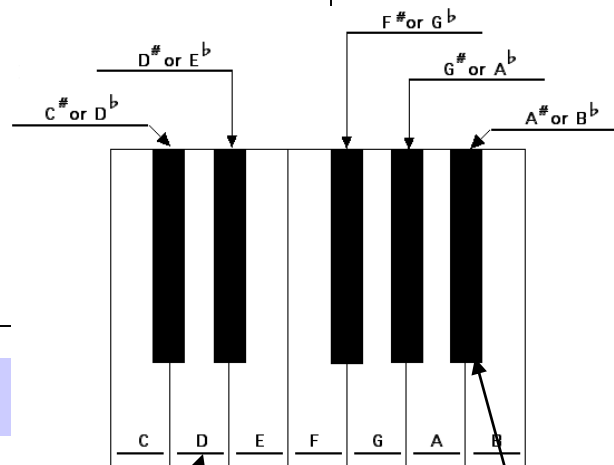
- ##
- #
- -
- b
- bb

$(7 \times 5) + 5$

C## / D / Ebb

D## / E / Fb

A# / Bb / Cbb



□ Fourth tier

- ####
- ##
- #
- -
- b
- bb
- bbb

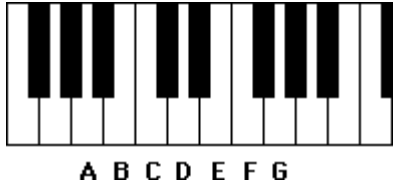
Base-40 Rule

Simple rule: Where a whole step exists between two key names, a null token is used.

Representation	Computation of Intervals			
CD01 = 1				
CD1 = 2				
CI = 3				
CD1 = 4				
CD#1 = 5				
- = 6				
DD01 = 7				
DD1 = 8				
DI = 9				
D#1 = 10				
D##1 = 11				
- = 12				
ED01 = 13				
ED1 = 14				
E1 = 15				
E#1 = 16				
E##1 = 17				
FD01 = 18				
FD1 = 19				
F1 = 20				
F#1 = 21				
F##1 = 22				
- = 23				
GD01 = 24				
GD1 = 25				
G1 = 26				
G#1 = 27				
G##1 = 28				
- = 29				
AD01 = 30				
A1 = 31				
A#1 = 32				
A##1 = 33				
- = 34				
- = 35				
BD01 = 36				
B01 = 37				
B1 = 38				
B#1 = 39				
B##1 = 40				

Interval	Delta	Interval	Delta
perfect unison	0	perfect octave	40
aug. unison	1	dim. octave	39
dim. second	4	aug. seventh	36
minor second	5	major seventh	35
major second	6	minor seventh	34
aug. second	7	dim. seventh	33
dim. third	10	aug. sixth	30
minor third	11	major sixth	29
major third	12	minor sixth	28
aug. third	13	dim. sixth	27
dim. fourth	16	aug. fifth	24
perfect fourth	17	perfect fifth	23
aug. fourth	18	dim. fifth	22

- The inversion of a simple interval is forty minus that interval.
- Intervals may be computed across the B - C octave boundary without extra calculations.
- Compound intervals such as tenths are related to intervals by the difference of an octave (40). A major tenth is 12 + 40 = 52.
- Limitations: Intervals involving not set, e.g. with three or more sharps or cannot be computed properly from this notation. Some unusual intervals which overlap the numbers for the standard intervals given above. For example, quadruple augmented unison between C#1 and C##1 has an interval value of 6, which is the number for a diminished second. Limitations can be removed by using a notation of a higher order.



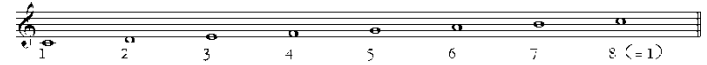
Example 4: "Seufzer, Tränen, Kummer, Not" from Cantata 21, *Ich hatte viel Bekümmern*



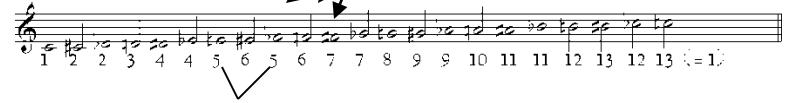
MIDI representation:	79	75	71	71	72	80	77	73	72	71
MIDI interval size:	4	4	0	1	8	3	4	1	1	
Base-40 representation:	186	174	158	158	163	191	180	168	163	158
Base-40 interval size:	12	16	0	5	28	11	12	5	5	
Standard interval name:	M3	d4	-	m2	m6	m3	M3	m2	m2	

From Base-40 to enharmonic preservation

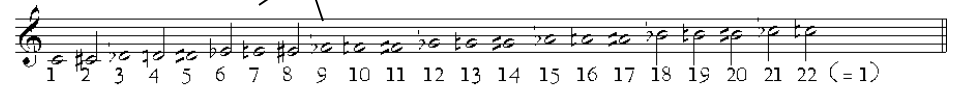
MIDI to base-7



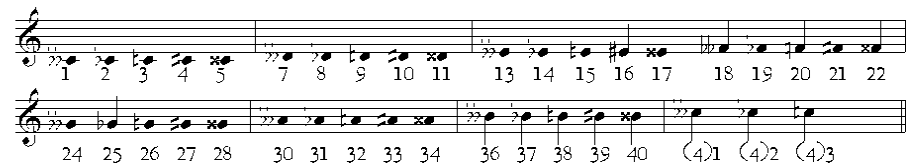
MIDI to base-12



MIDI to base-21



MIDI to base-40



Solution: Translate from symbolic code to MIDIPlus

What is MIDIPlus?

- In MIDI file format, a binary implementation of base-40
- Replaces last 3 bits of velocity byte
- Used to interpret key number

MIDIPLUS Correlation of Pitch Spelling to Specific MIDI Velocity Values

Value	Notated Pitch											
89	D $\flat\flat$	D \flat	E $\flat\flat$	F $\flat\flat$	F \flat	G $\flat\flat$	G \flat	A $\flat\flat$	A \flat	B $\flat\flat$	C $\flat\flat$	C \flat
90	C	C \sharp	D	E \flat	E	F	F \sharp	G	G \sharp	A	B \flat	B
91	B \sharp	B $\sharp\sharp$	C $\sharp\sharp$	D \sharp	D $\sharp\sharp$	E \sharp	E $\sharp\sharp$	F $\sharp\sharp$	F $\sharp\sharp\sharp$	G $\sharp\sharp$	A \sharp	A $\sharp\sharp$

Relevant handouts

Two translations of BWV 855 expressed with base-40

- E-Minor Fugue with enharmonically correct notation
 - http://esf.ccarh.org/MusicTheory_Tutorials/Base40_Handout_supp1.PDF
- E-Minor Fugue via MIDI-to-notation:
 - http://esf.ccarh.org/MusicTheory_Tutorials/Base40_Handout_supp2.PDF

Music theory tutorial:

http://esf.ccarh.org/MusicTheory_Tutorials/MusicTheory_ComputerApps.htm